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Joint Treatment of Random Variability and Imprecision in GPS Data Analysis

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Abstract. In the geodetic applications of the Global Positioning System (GPS) various types of data uncertainty are relevant. The most prominent ones are random variability (stochasticity) and imprecision. Stochasticity is caused by uncontrollable effects during the observation process. Imprecision is due to remaining systematic deviations between data and model due to imperfect knowledge or just for practical reasons. Depending on the particular application either stochasticity or imprecision may dominate the uncertainty budget. For the joint treatment of stochasticity and imprecision two main problems have to be solved. First, the imprecision of the original data has to be modelled in an adequate way. Then this imprecision has to be transferred to the quantities of interest. Fuzzy data analysis offers a proper mathematical theory to handle both problems. The main outcome is confidence regions for estimated parameters which are superposed by the effects of data imprecision. In the paper two applications are considered in a general way: the resolution of the phase ambiguity parameters and the estimation of point positions. The paper concludes with numerical examples for ambiguity resolution.

Key words: Fuzzy data analysis, imprecision, fuzzy confidence regions, GPS, ambiguity resolution

1 Introduction

Today, the Global Positioning System (GPS) is intensively used in geodetic applications as it is efficient and easy to access. The GPS consists of nominally 24 satellites on six orbital planes. It supplies the broadcast transmission of one-way microwave signals on two frequencies from the satellites to the individual ground stations. The 3D position of the GPS ground antenna and the receiver clock offset can be determined by simultaneously observing the signals of at least four satellites. This yields the satellite-receiver distances either directly using the code observations or indirectly via the (carrier wave) phase observations. For the second type of observations, the ambiguity parameters have to be determined. For further reading on GPS and on ambiguity resolution techniques see, e.g., Hofmann-Wellenhof et al. (1997), Parkinson and Spilker (1996), Teunissen and Kleusberg (1998).

GPS observations are biased by a variety of physical effects which have to be considered and handled in data processing. There are mainly three groups of causes. The most important one is due to the propagation of the signals. As the path of the GPS signals leads through the complete atmosphere, ionospherically and tropospherically caused travel-time delays have to be taken into account. They are superposed by multipath effects due to signal reflections in the vicinity of the tracking GPS antenna. The second group comprises all satellite effects like, e.g., signal transmission delays, satellite clock errors, satellite orbit errors, and satellite antenna offsets. Station and receiver effects like, e.g., signal reception delays and receiver clock errors belong to the third group. In addition, the GPS data processing results show characteristics due to the software and the operator.

Several techniques can be applied to reduce or eliminate most of the systematic effects such as the use of correction models with fixed or free parameters or of linear combinations of the GPS observations such as double differencing. Longer-term periodic signals such as diurnal ones can be weakened if the observation time is sufficiently long. However, such effects can not be eliminated completely due to the imperfect knowledge and the approximate character of the models in use, respectively. Hence, the uncertainty due to remaining systematic effects (imprecision) must be taken into account in addition to the random variability (stochasticity) of the observations. Kutterer (2001a, 2002) gives a general discussion of uncertainty in geodetic data analysis. Imprecision is particularly relevant in case of long distances between the GPS sites or very short observation intervals as in neither case it is possible to completely describe and remove systematic effects. This paper can be seen as an extension and generalization of the results given by Kutterer (2001b).

Fuzzy data analysis (Bandemer and Näther, 1992; Viertl, 1996) has proven to be an adequate mathematical tool to handle imprecision. Moreover, the combination of methods from stochastic and fuzzy theory allows the extension of classical geodetic data analysis to account for the effects due to superposed imprecision. In the following, the basics of fuzzy data analysis are presented. Two alternatives for the definition of fuzzy vectors are discussed. If the classical formulas of statistics are fuzzified by means of Zadeh's extension principle, stochasticity and imprecision can be treated simultaneously. Thus, imprecise confidence regions for the ambiguity parameters and for the point positions can be defined and discussed. At the end of the paper the results of simulation studies are given. They illustrate the applicability of the theory and quantify the impact of imprecision.

2 Basics of fuzzy data analysis

Fuzzy-theory was initiated by Zadeh (1965) in order to extend classical set theory by describing the degree (of membership) that a certain element belongs to a set. In classical set theory the membership degrees are either 1 (is element) or 0 (is not element). In fuzzy set theory the range of membership degree is [0,1]. Thus, a *fuzzy set* is defined as

$$\tilde{A} = \left\{ (x\,,\, m_{\tilde{A}}(x)) \,\middle|\, x \in X \right\}\,,\; m_{\tilde{A}}\,:\, X \to \left[0\,,\,1\right]. \qquad (1)$$

The degree of membership is given by the *membership function* which is denoted by $m_{\tilde{A}}(x)$. X is a classical set such as the set **R** of the real numbers. Important notions are the *support* of a fuzzy set (classical set with positive degrees of membership), the *height* of a fuzzy set (maximum membership degree), the *core* of a fuzzy set (the classical set with membership degree equal to 1), and the *α-cut* of a fuzzy set (classical set with membership degree greater equal $\alpha \in [0,1]$). For further reading see standard references on fuzzy data analysis such as Bandemer and Näther (1992) or Viertl (1996).

The most important operation in fuzzy-theory is the *intersection* of fuzzy sets. It is defined through the resulting membership function

$$m_{\tilde{A}\cap\tilde{B}} = \min\left(m_{\tilde{A}}\,,\, m_{\tilde{B}}\right) \qquad (2)$$

This definition is mostly used. Other consistent extensions of the classical intersection operator are available. See, e.g., Dubois and Prade (1980).

Fuzzy numbers can be defined based on fuzzy sets. A *fuzzy number* is a fuzzy set with a single element core and compact α-cuts. The *L-fuzzy numbers* defined by Dubois and Prade (1980) are widely used. They are exclusively considered in this paper. Their membership function is given by a strictly decreasing non-negative *reference function* L with [0,1] as the range of values.

$$m_{\tilde{X}}(x) = \begin{cases} L\left(\dfrac{x_m - x}{x_s}\right), & x_l \le x < x_m \\ L\left(\dfrac{x - x_m}{x_s}\right), & x_u \ge x \ge x_m \\ 0, & \text{else} \end{cases} \qquad (3)$$

Due to the single element core, L(0) = 1. For a graphical sketch of a L-fuzzy number with a linear reference function see Fig. 1. Formally, it can be represented by $\tilde{X} = (x_m,\, x_s)_L$. The *mean point* is denoted by x_m. The *spread* x_s serves as a scale factor. In practice, a typical membership function vanishes outside the interval given by the *lower bound* x_l and the *upper bound* x_u.

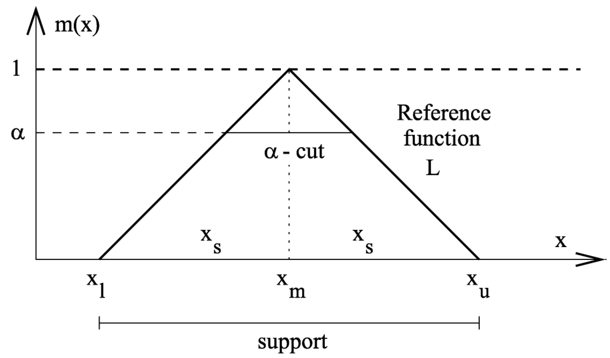

Fig. 1 L-fuzzy number with linear membership function (triangular fuzzy number)

The extension principle (Zadeh, 1965)

$$\tilde{B} = \tilde{g}(\tilde{A}_1, \ldots, \tilde{A}_n) :\Leftrightarrow$$

$$m_{\tilde{B}}(y) = \sup_{\substack{(x_1,\ldots,x_n)\in X_1\times\ldots\times X_n \\ g(x_1,\ldots,x_n)=y}} \min\Big((m_{\tilde{A}1}(x_1),\ldots,m_{\tilde{A}n}(x_n))\Big)\ \forall\ y \in Y \tag{4}$$

allows the generalization of functions with real arguments to functions with fuzzy arguments. For L-fuzzy numbers, the extended arithmetic rules are, e.g.,

$$\begin{aligned} \tilde{X} + \tilde{Y} &= (x_m + y_m, x_s + y_s)_L && \text{Addition} \\ \tilde{X} - \tilde{Y} &= (x_m - y_m, x_s + y_s)_L && \text{Subtraction} \\ a\tilde{X} &= (a\, x_m, |a|\, x_s)_L && \text{Multiplication by a real number} \end{aligned} \tag{5a, b, c}$$

The type of the reference function is preserved. The arithmetic operations can be carried out simply based on the mean points and the spreads. Please note that subtraction is not the inverse of addition. In fuzzy data analysis the spreads are regarded as measures of fuzziness or imprecision, respectively. Obviously, they are just added (linear propagation) in contrast to the addition of variances (quadratic propagation of the standard deviations) according to the Gaussian law of error propagation.

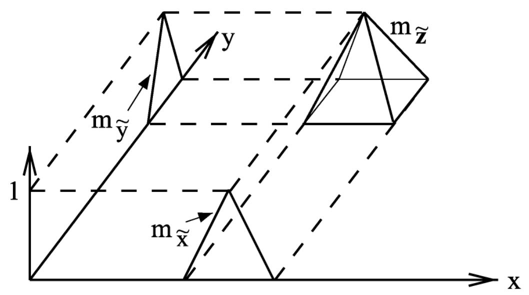

Fig. 2 Two-dimensional fuzzy vectors by the minimum rule

There are several possibilities to combine fuzzy numbers to a vector; see, e.g. Viertl (1996), Kutterer (2002). The mostly used way is to build a *fuzzy vector by the minimum rule*, i.e. using the minimum operator according to Eq. (2). In the 2D case this reads as

$$m_{\tilde{Z}}(\mathbf{z}) = \min(m_{\tilde{X}}(x), m_{\tilde{Y}}(y)) \tag{6}$$

For a graphical representation (linear reference function L as in Fig. 1) see Fig. 2. Such fuzzy vectors are called non-interactive (independent components).

A linear mapping of a fuzzy vector by the minimum rule can be approximated by the tightest inclusion

$$m_{\mathbf{F}\tilde{\mathbf{Z}}}(\mathbf{z}) = (\mathbf{F}\,\mathbf{z_m}\ , |\mathbf{F}|\,\mathbf{z_s})_L \tag{7}$$

The operation $|.|$ yields the matrix of the absolute values of the matrix components.

Interactive fuzzy vectors can be defined through

$$m_{\tilde{\mathbf{Z}}}(\mathbf{z}) = h\Big((\mathbf{z}-\mathbf{z_m})^T \mathbf{U}^{-1} (\mathbf{z}-\mathbf{z_m})\Big) \tag{8}$$

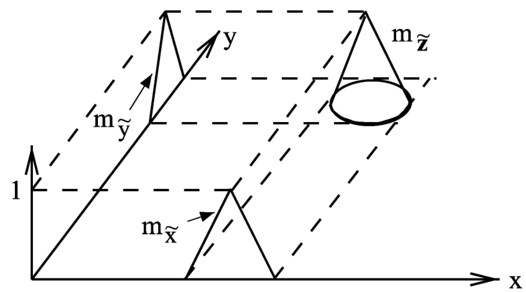

Fig. 3 Two-dimensional fuzzy vector of elliptic type

The function h is monotonously decreasing and non-negative with h(0)=1. The spreads and the interaction of the components are quantified in the positive definite *uncertainty matrix* **U**. Interaction is principally present due to the quadratic form which is the argument of h. Fuzzy vectors according to Eq. (8) are called *fuzzy vectors of elliptic type*. See Fig. 3 for a graphical representation; the function h of non-negative real arguments p and the matrix **U** are chosen as

$$h(p) = \max\big(1 - p^{1/2}, 0\big), \quad \mathbf{U} = \begin{bmatrix} x_s^{\ 2} & 0 \\ 0 & y_s^{\ 2} \end{bmatrix}$$

Linear mappings of fuzzy vectors of elliptic type are given in closed form by

$$m_{\tilde{\mathbf{Y}}=\mathbf{F}\tilde{\mathbf{Z}}}(\mathbf{y}) = h\Big((\mathbf{y}-\mathbf{y_m})^T \big(\mathbf{FUF}^T\big)^{-1} (\mathbf{y}-\mathbf{y_m})\Big) \tag{9}$$

The simplicity and closeness of Eqs. (8) and (9) is in contrast to the problems of motivating and formulating interactive (i.e. fuzzy-theoretically dependent) components. The equivalence with the Gaussian error propagation (variance propagation law) is obvious. But it has to be kept in mind that the interpretation is different since the membership functions must not be confused with the independently defined density functions of probability theory. Nevertheless, a quadratic propagation of spreads is available by means of fuzzy vectors of elliptic type; see Eq. (9).

3 Modeling and propagation of data imprecision

Fuzziness and imprecision are considered as being identical in the following as it is common practice in fuzzy data analysis. Hence, fuzzy data analysis can be applied to handle the impact of observation imprecision on the parameters of interest. As already motivated in Section 1, there are several sources of imprecision in GPS data acquisition and analysis. Hence, both stochasticity and imprecision have to be considered in a general combined approach. Stochasticity is assumed to be superposed by imprecision. This is the basic condition of the extension principle according to Eq. (4).

There are three steps to derive the imprecision of the quantities of interest. First, the imprecision of a single observation has to be described by means of a fuzzy (or imprecise) number. This can be based on a questionnaire to be completed by experts in order to assess the particular application; see, e.g., Kutterer (2002) for details. Second, the fuzzy numbers representing the imprecise observations have to be combined to a fuzzy (or imprecise) vector. This can be based on the two types of fuzzy vectors given in Section 2; see Eq. (6) for the definition of a fuzzy vector by the minimum rule and Eq. (8) for a fuzzy vector of elliptic type. Third, the extension principle according to Eq. (4) has to be applied to the real-valued functional expressions. Here, the least-squares estimator (LSE)

$$\hat{\boldsymbol{\beta}} = \left(\mathbf{X}^{\mathrm{T}}\mathbf{W}\mathbf{X}\right)^{\mathbf{-1}}\mathbf{X}^{\mathrm{T}}\mathbf{W}\,\mathbf{y} \qquad (10)$$

of the (deterministic) parameters $\boldsymbol{\beta}$ in a Gauss-Markoff model is considered first. Its variance-covariance matrix (vcm) reads as

$$\boldsymbol{\Sigma}_{\hat{\beta}\hat{\beta}} = \sigma_0^2\left(\mathbf{X}^{\mathrm{T}}\mathbf{W}\mathbf{X}\right)^{-1} \qquad (11)$$

The column-regular [n×u]-dimensional configuration matrix is denoted by $\mathbf{X}$ and the [n×n]-dimensional regular weight matrix of the observations by $\mathbf{W}$. The vector of the observations is represented by $\mathbf{y}$. The a priori variance factor is given by σ_0^2.

The second quantity of interest is the (1-γ)-confidence region für the expected value μ of $\hat{\beta}$ which is given by

$$K_{1-\gamma}\left(\hat{\boldsymbol{\beta}}\right) = \left\{ \boldsymbol{\mu} \,\middle|\, \left(\boldsymbol{\mu}-\hat{\boldsymbol{\beta}}\right)^{\mathrm{T}} \boldsymbol{\Sigma}_{\hat{\beta}\hat{\beta}}^{-1}\left(\boldsymbol{\mu}-\hat{\boldsymbol{\beta}}\right) \le \chi^2_{u,1-\gamma} \right\} \qquad (12)$$

with $\chi^2_{n,1-\gamma}$ the (1-γ)-fractile value of the χ^2–distribution with u degrees of freedom.

3.1 Extended least-squares estimator

In the first case (LSE according to Eq. (10)) the extension principle reads as

$$m_{\tilde{\hat{\boldsymbol{\beta}}}}\left(\hat{\boldsymbol{\beta}}\right) = \sup_{\substack{\mathbf{y}\in R^n \\ \hat{\beta}=\left(\mathbf{X}^{\mathrm{T}}\mathbf{W}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{W}\mathbf{y}}} m_{\tilde{\mathbf{y}}}(\mathbf{y}) \quad \forall\, \hat{\boldsymbol{\beta}} \in R^u \qquad (13)$$

with $m_{\tilde{\mathbf{y}}}(\mathbf{y})$ the membership function of the vector of the n imprecise observations and $m_{\tilde{\hat{\beta}}}\left(\hat{\boldsymbol{\beta}}\right)$ the membership function of the vector of the u imprecise estimated parameters. The use of fuzzy vectors by the minimum rule yields

$$m_{\tilde{\hat{\beta}}}\left(\hat{\boldsymbol{\beta}}\right) = \left(\left(\mathbf{X}^{\mathrm{T}}\mathbf{W}\mathbf{X}\right)^{\mathbf{-1}}\mathbf{X}^{\mathrm{T}}\mathbf{W}\,\mathbf{y}_m, \left|\left(\mathbf{X}^{\mathrm{T}}\mathbf{W}\mathbf{X}\right)^{\mathbf{-1}}\mathbf{X}^{\mathrm{T}}\mathbf{W}\right|\mathbf{y}_s\right)_L \qquad (14)$$

according to Eq. (7). The use of fuzzy vectors of elliptic type yields

$$m_{\tilde{\hat{\beta}}}\left(\hat{\boldsymbol{\beta}}\right) = h\left(\left(\hat{\boldsymbol{\beta}}-\hat{\boldsymbol{\beta}}_{\mathbf{m}}\right)^{\mathrm{T}}\left(\left(\mathbf{X}^{\mathrm{T}}\mathbf{W}\mathbf{X}\right)^{\mathbf{-1}}\mathbf{X}^{\mathrm{T}}\mathbf{W}\,\mathbf{U}\,\mathbf{W}\mathbf{X}\left(\mathbf{X}^{\mathrm{T}}\mathbf{W}\mathbf{X}\right)^{\mathbf{-1}}\right)^{-1}\left(\hat{\boldsymbol{\beta}}-\hat{\boldsymbol{\beta}}_{\mathbf{m}}\right)\right)$$

according to Eq. (9) with

$$\hat{\boldsymbol{\beta}}_m = \left(\mathbf{X}^{\mathrm{T}}\mathbf{W}\mathbf{X}\right)^{\mathbf{-1}}\mathbf{X}^{\mathrm{T}}\mathbf{W}\,\mathbf{y}_m$$

This can be rewritten as

$$m_{\tilde{\hat{\beta}}}\left(\hat{\boldsymbol{\beta}}\right) = h\left(\left(\hat{\boldsymbol{\beta}}-\hat{\boldsymbol{\beta}}_{\mathbf{m}}\right)^{\mathrm{T}}\mathbf{U}_{\hat{\beta}\hat{\beta}}^{-1}\left(\hat{\boldsymbol{\beta}}-\hat{\boldsymbol{\beta}}_{\mathbf{m}}\right)\right) \qquad (15a)$$

with the imprecision matrix

$$\mathbf{U}_{\hat{\beta}\hat{\beta}} = \left(\mathbf{X}^{\mathrm{T}}\mathbf{W}\mathbf{X}\right)^{\mathbf{-1}}\mathbf{X}^{\mathrm{T}}\mathbf{W}\,\mathbf{U}\,\mathbf{W}\mathbf{X}\left(\mathbf{X}^{\mathrm{T}}\mathbf{W}\mathbf{X}\right)^{\mathbf{-1}} \qquad (15b)$$

respectively; see, e.g., Kutterer (2002). Please note that for both the fuzzy vectors by the minimum rule and the fuzzy vectors of elliptic type the mean point vectors $\hat{\beta}_m$ are identical with the classical (precise) least-squares estimators. Thus, both presented fuzzy extensions are consistent with the real-valued case.

3.2 Extended confidence regions

In case of the confidence regions, see Eq. (12), the extension principle reads as

$$m_{\tilde{K}_{1-\gamma}}\left(\hat{\beta}\right) = \sup_{\mu \in K_{1-\gamma}\left(\hat{\beta}\right)} m_{\tilde{\beta}}\left(\mu\right) \quad \forall \quad \hat{\beta} \in R^{u} \tag{16}$$

Eq. (16) represents a constrained optimization problem. The imprecise confidence region is the solution of this problem. In order to obtain a closed-form expression for the results, fuzzy vectors of elliptic type

$$m_{\tilde{\beta}}(\mu) = h\left(\left(\mu - \hat{\boldsymbol{\beta}}_{\mathbf{m}}\right)^{T} \mathbf{U}_{\hat{\beta}\hat{\beta}}^{-1}\left(\mu - \hat{\boldsymbol{\beta}}_{\mathbf{m}}\right)\right) \tag{17}$$

as given in Eqs. (15a, b) are solely considered in the following. The function h is strictly decreasing for non-negative arguments. Hence, the supremum or maximum functional value, respectively, is obtained with the minimum argument value (quadratic Euclidean distance with respect to $\mathbf{U}_{\hat{\beta}\hat{\beta}}$)

$$d^{2}_{2,\mathbf{U}_{\hat{\beta}\hat{\beta}}}\left(\mu, \hat{\boldsymbol{\beta}}_{\mathbf{m}}\right) = \left(\mu - \hat{\boldsymbol{\beta}}_{\mathbf{m}}\right)^{T} \mathbf{U}_{\hat{\beta}\hat{\beta}}^{-1}\left(\mu - \hat{\boldsymbol{\beta}}_{\mathbf{m}}\right)$$

under the side condition

$$\mu \in K_{1-\gamma}\left(\hat{\beta}\right) = \left\{ \mu \left| \left(\mu - \hat{\beta}\right)^{T} \Sigma_{\hat{\beta}\hat{\beta}}^{-1}\left(\mu - \hat{\beta}\right) \le \chi^{2}_{u,1-\gamma} \right. \right\}$$

An equivalent side condition is

$$\left(\mu - \hat{\beta}\right)^{T} \Sigma_{\hat{\beta}\hat{\beta}}^{-1}\left(\mu - \hat{\beta}\right) = \kappa^{2}, \text{ with } \kappa^{2} \in \left[0, \chi^{2}_{u,1-\gamma}\right]$$

Thus, the objective function to be minimized with respect to (w.r.t.) μ reads as

$$\Phi\left(\mu\right) = \left(\mu - \hat{\boldsymbol{\beta}}_{\mathbf{m}}\right)^{T} \mathbf{U}_{\hat{\beta}\hat{\beta}}^{-1}\left(\mu - \hat{\boldsymbol{\beta}}_{\mathbf{m}}\right) - \lambda\left(\left(\mu - \hat{\beta}\right)^{T} \Sigma_{\hat{\beta}\hat{\beta}}^{-1}\left(\mu - \hat{\beta}\right) - \kappa^{2}\right), \text{ with } \kappa^{2} \in \left[0, \chi^{2}_{u,1-\gamma}\right] \tag{18}$$

and with the Lagrangian multiplier λ.

Two special cases can be distinguished: Obviously, as long as $\hat{\beta} \in K_{1-\gamma}\left(\hat{\beta}_m\right)$, there is always a $\mu \in K_{1-\gamma}\left(\hat{\beta}\right)$ with $\mu = \hat{\beta}_m$ and hence $d^{2}_{2,\mathbf{U}_{\hat{\beta}\hat{\beta}}}\left(\mu, \hat{\boldsymbol{\beta}}_{\mathbf{m}}\right) = 0$. Consequently, the resulting value of the membership function is equal to one. Hence, the obtained classical set corresponds to the confidence region given by Eq. (12).

In all other cases, i.e., $\hat{\beta} \notin K_{1-\gamma}\left(\hat{\beta}_m\right)$, there is $\kappa^{2} = \chi^{2}_{u,1-\gamma} = constant$. Then the problem can be understood in a geometrical way as the determination of the distance between the point $\hat{\beta}_m$ and the hyperellipsoid $K_{1-\gamma}\left(\hat{\beta}\right) = \left\{ \mu \left| \left(\mu - \hat{\beta}\right)^{T} \Sigma_{\hat{\beta}\hat{\beta}}^{-1}\left(\mu - \hat{\beta}\right) = \chi^{2}_{u,1-\gamma} \right. \right\}$. As now the distance is in any case positive, the corresponding values of the membership function are less than one. Hence, the confidence region according to Eq. (12) is the core (see Section 2) of the extended confidence region.

The objective function given in Eq. (18) now reads as

$$\Phi\left(\mu\right) = \left(\mu - \hat{\boldsymbol{\beta}}_{\mathbf{m}}\right)^{T} \mathbf{U}_{\hat{\beta}\hat{\beta}}^{-1}\left(\mu - \hat{\boldsymbol{\beta}}_{\mathbf{m}}\right) - \lambda\left(\left(\mu - \hat{\beta}\right)^{T} \Sigma_{\hat{\beta}\hat{\beta}}^{-1}\left(\mu - \hat{\beta}\right) - \kappa^{2}\right), \text{ with } \kappa^{2} = \chi^{2}_{u,1-\gamma} \tag{19}$$

The determination of the stationary point which refers to the minimum requires the differentiation of Φ w.r.t. μ which yields

$$\frac{\partial \Phi\left(\mu\right)}{\partial \mu} = 2\left(\mu - \hat{\boldsymbol{\beta}}_{\mathbf{m}}\right)^{T} \mathbf{U}_{\hat{\beta}\hat{\beta}}^{-1} - 2\lambda\left(\mu - \hat{\beta}\right)^{T} \Sigma_{\hat{\beta}\hat{\beta}}^{-1} = \mathbf{0}$$

and

$$\mathbf{U}_{\hat{\beta}\hat{\beta}}^{-1}\left(\mu - \hat{\boldsymbol{\beta}}_{\mathbf{m}}\right) - \lambda\, \Sigma_{\hat{\beta}\hat{\beta}}^{-1}\left(\mu - \hat{\beta}\right) = 0\,. \tag{20}$$

Hence,

$$\mu - \hat{\beta} = \left(\mathbf{U}_{\hat{\beta}\hat{\beta}}^{-1} - \lambda\, \Sigma_{\hat{\beta}\hat{\beta}}^{-1}\right)^{-1} \mathbf{U}_{\hat{\beta}\hat{\beta}}^{-1}\left(\hat{\beta}_m - \hat{\beta}\right) \tag{21}$$

Differentiation of Φ w.r.t. λ yields

$$\left(\mu - \hat{\beta}\right)^{T} \Sigma_{\hat{\beta}\hat{\beta}}^{-1}\left(\mu - \hat{\beta}\right) - \kappa^{2} = 0.$$

Insertion of Eq. (21) into the last one finally yields the single equation

$$\left(\hat{\beta}_m-\hat{\beta}\right)^T\left(\Sigma_{\hat{\beta}\hat{\beta}}-\lambda\ \mathbf{U}_{\hat{\beta}\hat{\beta}}\right)^{-1}\Sigma_{\hat{\beta}\hat{\beta}}\left(\Sigma_{\hat{\beta}\hat{\beta}}-\lambda\ \mathbf{U}_{\hat{\beta}\hat{\beta}}\right)^{-1}\left(\hat{\beta}_m-\hat{\beta}\right)=\kappa^2$$

which is nonlinear w.r.t. the single unknown λ. λ can be determined numerically by a common root-finding method. When its actual value is known, Eqs. (20) and (21) yield

$$\mu-\hat{\beta}_{\mathbf{m}}=\lambda\ \mathbf{U}_{\hat{\beta}\hat{\beta}}\ \Sigma^{-1}_{\hat{\beta}\hat{\beta}}\left(\mu-\hat{\beta}\right)=\lambda\ \mathbf{U}_{\hat{\beta}\hat{\beta}}\ \Sigma^{-1}_{\hat{\beta}\hat{\beta}}\left(\mathbf{U}^{-1}_{\hat{\beta}\hat{\beta}}-\lambda\ \Sigma^{-1}_{\hat{\beta}\hat{\beta}}\right)^{-1}\mathbf{U}^{-1}_{\hat{\beta}\hat{\beta}}\left(\hat{\beta}_m-\hat{\beta}\right) \tag{22}$$

Finally the membership function of the imprecise confidence region is obtained regarding Eq. (16) as

$$m_{\tilde{K}_{1-\gamma}}(\hat{\beta}) = \begin{cases} 1, & \hat{\beta}\in K_{1-\gamma}\left(\hat{\beta}_m\right) \\ h\left(\left(\mu-\hat{\beta}_{\mathbf{m}}\right)^T\mathbf{U}^{-1}_{\hat{\beta}\hat{\beta}}\left(\mu-\hat{\beta}_{\mathbf{m}}\right)\right), & \hat{\beta}\notin K_{1-\gamma}\left(\hat{\beta}_m\right) \end{cases}. \tag{23}$$

with $\left(\mu-\hat{\beta}_{\mathbf{m}}\right)$ as given in Eq. (22).

Please note that the second derivative of Φ w.r.t. μ

$$\frac{\partial^2\Phi(\mu)}{\partial\mu^2}=2\left(\mathbf{U}^{-1}_{\hat{\beta}\hat{\beta}}-\lambda\ \Sigma^{-1}_{\hat{\beta}\hat{\beta}}\right)$$

has to be positive definite to assure a minimum what can easily be checked.

Eq. (23) describes the extension of classical confidence regions to confidence regions where the originally random-type quantities of interest are superposed by imprecision. It is a significant generalization of the corresponding Eq. (20) in Kutterer (2001b) where $\mathbf{U}_{\hat{\beta}\hat{\beta}}$ and $\Sigma_{\hat{\beta}\hat{\beta}}$ had to be proportional. Like in the analysis of GPS observations both the phase ambiguity search spaces and the precision of point positions are represented by confidence regions, Eq. (23) plays the key role in any case when imprecision has to be taken into account.

Fig. 4 shows exemplarily for the 2D case the superposition of a classical (1-γ) confidence ellipse and an imprecise 2D vector of elliptic type. It is obvious that the superposition of the two quantities does not yield an elliptic quantity. The maximum membership degree is obtained for the mean point of the imprecise vector and the corresponding confidence ellipse. This is only valid for the classical confidence region.

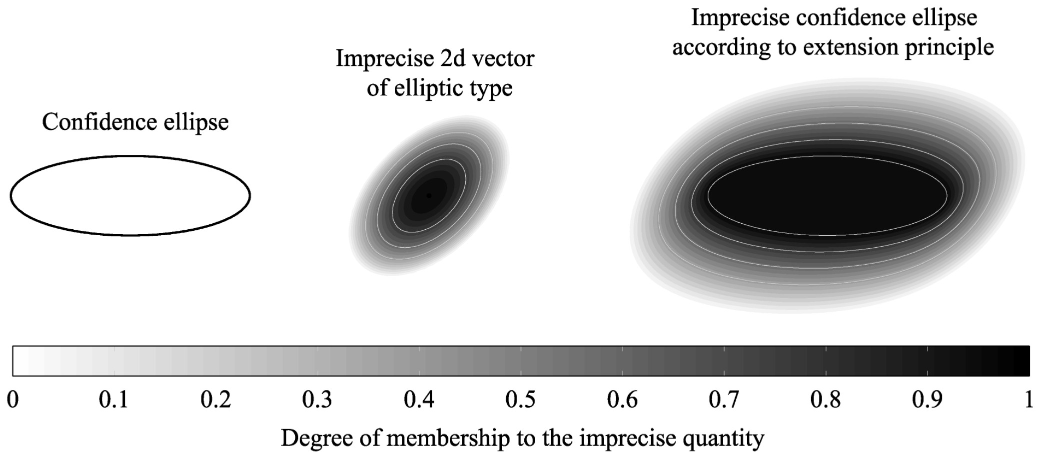

Fig. 4 Imprecise confidence ellipse as resulting from the superposition of a classical (precise) confidence ellipse and an imprecise vector. The quantities are placed separately for the sake of better representation. The light-gray lines indicate isolines of the membership values.

The qualititative difference of the presented results from the common unterstanding of accuracy is obvious from the extended LSE and the extended confidence regions. Actually, the introduced combined measures of stochasticity and imprecision are closer to the idea of accuracy in practical applications. The classical statistical point of view implies reduction of uncertainty just by repetition of observations. If fuzzy-theory is used to model and handle imprecision this is not possible. The amount of imprecision is kept when observations are repeated. Imprecision can only be reduced outside the particular observation scenario as it is according to common sense.

4 Extended GPS phase ambiguity search spaces

The linearized functional model of GPS code and phase observations reads as

$$E(\mathbf{y}) = \mathbf{X}\,\boldsymbol{\beta} = \mathbf{A}\,\boldsymbol{\xi} + \mathbf{Z}\,\boldsymbol{\zeta} \tag{24}$$

with the expectation E(.), the real-valued parameters ξ such as coordinates and the integer ambiguity parameters ζ. The matrices **A** and **Z** denote the two corresponding components of the configuration matrix **X**. Please note that the vector **y** comprises both code and phase observations or differences, respectively. For the following there is no need to distinguish between undifferenced and double-differenced observations. The only impact is then on the adequate parametrization. The rows of matrix **Z** which correspond with the code observations are naturally equal to zero. The vcm or dispersion matrix of **y** given by

$$D(\mathbf{y}) = \boldsymbol{\Sigma}_{\mathbf{yy}} \tag{25}$$

A real-valued approximation of the integer ambiguity parameters is obtained by a least-squares estimation weighted by $\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}$ as

$$\hat{\boldsymbol{\zeta}} = \mathbf{F}\,\mathbf{y} \tag{26a}$$

with

$$\mathbf{F} = \left(\mathbf{Z}^{T}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\left(\boldsymbol{\Sigma}_{\mathbf{yy}} - \mathbf{A}\left(\mathbf{A}^{T}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\mathbf{A}\right)^{-1}\mathbf{A}^{T}\right)\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\mathbf{Z}\right)^{-1} \times \left(\mathbf{Z}^{T}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\left(\boldsymbol{\Sigma}_{\mathbf{yy}} - \mathbf{A}\left(\mathbf{A}^{T}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\mathbf{A}\right)^{-1}\mathbf{A}^{T}\right)\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\right) \tag{26b}$$

what leads to

$$\boldsymbol{\Sigma}_{\hat{\zeta}\hat{\zeta}} = \mathbf{F}\,\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\,\mathbf{F}^{T} = \left(\mathbf{Z}^{T}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\left(\boldsymbol{\Sigma}_{\mathbf{yy}} - \mathbf{A}\left(\mathbf{A}^{T}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\mathbf{A}\right)^{-1}\mathbf{A}^{T}\right)\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\mathbf{Z}\right)^{-1} \tag{27}$$

for the vcm of the real-valued estimates of the integer ambiguity parameters. Consequently, the corresponding (1-γ)- confidence hyperellipsoid regarding Eq. (12) reads as

$$K_{1-\gamma}\left(\hat{\zeta}\right) = \left\{\mu \,\middle|\, \left(\mu - \hat{\zeta}\right)^{T}\boldsymbol{\Sigma}_{\hat{\zeta}\hat{\zeta}}^{-1}\left(\mu - \hat{\zeta}\right) \le \chi^{2}_{f,1-\gamma}\right\} \tag{28}$$

with f denoting the number of ambiguity parameters; see Kutterer (2001b). The confidence region given in Eq. (28) can be set up based on code observations only. It serves as a search space for the integer ambiguity parameters.

The methods proposed in literature for ambiguity resolution differ mainly in the strategy how to identify the „correct“ ambiguity parameter. In any case they depend and rely on the adequateness of the models given in Eqs. (24) and (25). In particular, the functional model has to be accurate in the meaning that the existing errors are only assignable to the observations and that they are all and exclusively random. However, this does not hold in general. This assumption is certainly not suitable for short observation times or real-time applications and for long baselines. Hence, the imprecision of the observations has to be assessed and modelled as mentioned above. Then it has to be superposed to the search space by applying the procedure shown in Section 3, in particular by using Eq. (23).

As the extended search space is obviously enlarged, more candidate vectors have to be taken into account for ambiguity resolution. If a rounding procedure is applied such as the LAMBDA method (Teunissen and Kleusberg, 1998), there is no change for the integer-estimated ambiguity vector. However, due to the increased number of candidates the separability of the best and the second-best solution may be reduced which leads to more reliable results. In all other methods like, e.g., On-The-Fly algorithms (Abidin, 1993; Leinen, 2001), the degree of imprecision given by the membership function of the extended search space offers additional information for the validity of the solution.

5 Extended error measures for GPS site positions

Imprecise (1-γ)-confidence regions (ellipses and ellipsoids, respectively) for the 3D positions of GPS sites can be given in analogy to the ambiguity resolution presented in Section 4. As soon as the ambiguity parameters are known (and fixed, respectively), the phase observations can be used as highly precise distance observations. The functional model according to Eq. (23) simplifies to

$$E(\bar{\mathbf{y}}) = \mathbf{A}\,\boldsymbol{\xi},\ \text{with}\ \bar{\mathbf{y}} = \mathbf{y} - \mathbf{Z}\,\boldsymbol{\zeta}, \tag{29}$$

but the stochastic model represented by the vcm

$$D(\bar{\mathbf{y}}) = \boldsymbol{\Sigma}_{\bar{\mathbf{y}}\bar{\mathbf{y}}} = \boldsymbol{\Sigma}_{\mathbf{yy}} \tag{30}$$

is unchanged because the introduced ambiguities are considered as exact. Least-squares estimation of the remaining real-valued unknown parameters like, e.g., position coordinates or tropospheric parameters, yields

$$\hat{\boldsymbol{\xi}} = \left(\mathbf{A}^{T}\boldsymbol{\Sigma}_{\bar{\mathbf{y}}\bar{\mathbf{y}}}^{-1}\mathbf{A}\right)^{-1}\mathbf{A}^{T}\boldsymbol{\Sigma}_{\bar{\mathbf{y}}\bar{\mathbf{y}}}^{-1}\bar{\mathbf{y}} \tag{31}$$

with the corresponding vcm

$$\boldsymbol{\Sigma}_{\hat{\xi}\hat{\xi}} = \left(\mathbf{A}^{\mathrm{T}}\, \boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\, \mathbf{A}\right)^{-1} \tag{32}$$

The submatrix for a set of parameters such as the coordinates of a particular point is obtained by means of a selection matrix like, e.g.,

$$\mathbf{S}_{\mathrm{i}} = \left[\mathbf{0}_{3\times 3(\mathrm{i}-1)} \quad \mathbf{I}_3 \quad \mathbf{0}_{3\times(\mathrm{u}-3\mathrm{i})}\right] \tag{33}$$

Hence, for the position of the i^{th} point it is

$$\hat{\xi}_{\mathrm{i}} = \mathbf{S}_{\mathrm{i}} \left(\mathbf{A}^{\mathrm{T}}\, \boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\, \mathbf{A}\right)^{-1} \mathbf{A}^{\mathrm{T}}\, \boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\, \overline{\mathbf{y}} \tag{34}$$

and

$$\boldsymbol{\Sigma}_{\hat{\xi}\hat{\xi},\mathrm{i}} = \mathbf{S}_{\mathrm{i}} \left(\mathbf{A}^{\mathrm{T}}\, \boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\, \mathbf{A}\right)^{-1} \mathbf{S}_{\mathrm{i}}^{\mathrm{T}} \tag{35}$$

Its classical (1-γ)-confidence ellipsoid reads as

$$\mathrm{K}_{1-\gamma}\left(\hat{\xi}_{\mathrm{i}}\right) = \left\{ \mu \,\middle|\, \left(\mu - \hat{\xi}_{\mathrm{i}}\right)^{\mathrm{T}} \boldsymbol{\Sigma}_{\hat{\xi}\hat{\xi},\mathrm{i}}^{-1} \left(\mu - \hat{\xi}_{\mathrm{i}}\right) \le \chi^2_{3,1-\gamma} \right\}$$

The corresponding imprecise confidence ellipsoid is obtained by means of the procedure given in Section 3, mainly using Eq. (23).

6. Examples

In the following, the impact of the proposed superposition of stochasticity and imprecision on the size of the search space is shown exemplarily. The main idea is to extend the classical search space by scaling the semi-axes of the respective confidence hyperellipsoid so that the result is the tightest inclusion of the support of the imprecise confidence region. From a practical point of view this is an important first step to consider imprecise observations. Below, several GPS real-time scenarios are simulated and discussed. This section is organized as follows. First, the general configuration and the estimation procedure are given. Second, the modeling of the imprecise observations and the derivation of the imprecise vector of the ambiguity parameters are described. Third, the resulting scaling factors for the classical search spaces are compiled and discussed. The results of the simulation runs were derived by means of the procedure which was described in Section 3 and which led to Eq. (23).

The scenarios are based on the nominal GPS configuration with 24 satellites which was simulated according to orbital elements published by Parkinson and Spilker (1996). The respective solutions are based on single epoch observations to all visible satellites. The number of satellites was controlled by means of an elevation mask: If n satellites were visible and m < n satellites were considered, those n-m satellites with lowest elevation were dropped. The (1-γ)-confidence hyperellipsoids are based on a code-only approximate position. The standard deviation of the code observations was chosen as 0.3 m in order to obtain realistic magnitudes. The integer ambiguity parameters were approximated by means of a least-squares adjustment (float solution) as it is common practice in GPS data analysis. From Fig. 4 it is already clear that the resulting imprecise confidence regions are no hyperellipsoids but more complex quantities. They are fuzzy supersets of the classical precise hyperellipsoids.

The actual ratio of stochasticity and imprecision of the observations depends on the respective configuration. It is part of the complete uncertainty budget. The relevant types of uncertainty can be described and quantified by experts in several ways using a detailed questionnaire: A sensitivity analysis of the applied correction models for example gives insight in critical parameters, site-dependent effects such as multipath can be assessed by studying the local situation, and extensive controlled variations of the observation configuration indicate the magnitude of external effects.

In the following examples some illustrative values were chosen for the amount of imprecision. The imprecise vector of the real-valued approximation of the ambiguity parameters was represented as an imprecise vector of elliptic type. It was deduced from its range of values (convex polyhedron) which is directly computable from the imprecision of the observations in the considered cases because of the relatively low dimension of the parameter space. This polyhedron was then enclosed by the tightest possible hyperellipsoid in order to define the support of an imprecise vector of elliptic type. A linear reference function was chosen as in Figures 1 and 3 and Eq. (3), respectively.

For the first simulation runs the imprecision of all code observations was introduced as 0.03 m (10% of the value of the standard deviation) what is a very restrained assumption. Table 1 shows the scaling factors of the semi-axes of the classical search space which were obtained for GPS observation sites in three different latitudes: equatorial region (Latitude = 0°), mean latitudes (45°) and the poles (90°). It is obvious that the scaling factor depends only slightly on the configuration - less on to the latitude and more on the number of satellites. The latter is due to the fact that the amount of imprecision increases by the number of observations. There is no significant dependence of the results on the time of observation. By taking an average value of 1.25 one can state that the assumed imprecision of 10% requires an extension of the search space by 25%.

Tab. 1 Maximum scaling factor for the semi-axes of the classical (precise) ambiguity search space in the real-time case (single epoch observations) to take imprecision into account. The standard deviations of the code observations equal 0.3 m, the imprecision equals 0.03 m.

Latitude → # of sat. ↓	0°	45°	90°
4	1.20	1.21	1.20
5	1.23	1.23	1.23
6	1.25	1.25	1.25
7	1.25	1.26	1.27
8	1.27	1.27	1.27
9	1.29	—	1.28

In further simulations runs the ratio of imprecision and stochasticity (in terms of standard deviations) was varied. Table 2 shows the results which were derived for the GPS observation site with latitude = 45° for different ratios. A linear dependence of the scaling factor on the chosen ratio can be found. In case of identical magnitudes of stochasticity and imprecision (ratio=1) the semi-axes of the search spaces need to be increased by a factor significantly larger than 3.

Tab. 2 Maximum scaling factor for the semi-axes of the classical (precise) ambiguity search space in the real-time case (single epoch observations) to take imprecision into account. The standard deviations of the code observations equals 0.3 m, the imprecision is varied.

% of std.dev. → # of sats. ↓	$0.1\cdot\sigma_{code}$	$0.2\cdot\sigma_{code}$	$0.5\cdot\sigma_{code}$	$1.0\cdot\sigma_{code}$	$2.0\cdot\sigma_{code}$
4	1.21	1.41	2.03	3.06	5.12
5	1.23	1.47	2.17	3.33	5.65
6	1.25	1.51	2.26	3.52	6.05
7	1.26	1.52	2.31	3.61	6.23
8	1.27	1.54	2.35	3.71	6.41

The quality of the approximation of the actual imprecise search space by scaling the classical (precise) one becomes poorer with increasing importance of imprecision. In such cases the individual scaling factors for the respective semi-axes can differ significantly. Fig. 4 illustrates this: The imprecise confidence ellipse is principally obtained by superposing two ellipses; however, the resulting quantity is not elliptic and cannot be represented uniquely by an ellipse. Nevertheless, if the maximum values for the scaling factors are taken the inclusion property is kept in any case.

The examples indicate that the ratio of stochasticity and imprecision plays a leading role in the extension of the classical search space in order to take imprecision into account. Hence, the traditional procedure of ambiguity resolution is inadequate when imprecision dominates the uncertainty budget. A rule-of-thumb for practitioners reads as follows: The search space needs to be extended even in the case of low imprecision. When the observations are likely to be imprecise at least in the same magnitude as stochasticity the semi-axes of the search space should be lengthened by a factor of at least 3. In this way the quality of the validation of the resolved ambiguity vector can be improved. Some remarks on this topic were also made at the end of Section 4.

7 Conclusions and outlook

As imprecision has to be considered in a variety of geodetic applications of the GPS the joint treatment of stochasticity and imprecision in GPS data analysis is important. Imprecision is an independent type of uncertainty and in general it can not be reduced or transformed to stochasticity. Hence, the common modeling as given in Eqs. (24) and (25) is incomplete because it does not take imprecision into account. Fuzzy-theory allows to distinguish strictly between these two types of uncertainty. For this reason it is suitable to handle both stochasticity and imprecision. Moreover, it allows to control the type of propagation of imprecision from the observations to the parameters; see Eq. (7) for linear propagation and Eq. (9) for quadratic propagation, respectively. In both cases the classical least-squares estimator is kept as mean point of the resulting fuzzy set.

The benefit of the joint treatment of stochasticity and imprecision for the GPS community is two-fold. On the one hand the resolution of the phase ambiguity parameters can be improved by extending the classical search space by simple scaling. This is a first step to more reliable results in real-time GPS. On the other hand the quality of the point positions determined by means of GPS can be described more thoroughly. It is well known that their formal precision is too optimistic. Imprecise confidence regions can objectivy the common measures of precision and accuracy since imprecision cannot be reduced by repeated observations.

There are some issues which are worthwhile for further studies. First, the uncertainty budget of GPS observation configurations has to be evaluated thoroughly for the practical application of the presented approach. Thus, a look-up table for observation imprecision could be worked out for typical configurations. Second, the notion of imprecision was based here on L-fuzzy numbers which imply identical left and right spreads. In a more general formulation LL-fuzzy numbers can be used which have identical left and right reference functions but different spreads; see standard references on fuzzy-theory. In this way the knowledge of possible asymmetries in remaining systematic effects could be modeled what would lead directly to biases in least-squares estimation which have to be taken into account. Third, it is up to now not sufficiently understood how imprecision propagates in practice from the observations to the parameters of interest. There could be more possibilities than the linear and the quadratic propagation which were considered in this paper.

References

Abidin H. (1993): ***Computational and Geometrical Aspects of On-The-Fly Ambiguity Resolution***. Ph.D. dissertation. Department of Surveying Engineering Technical Report No. 164, University of New Brunswick, Fredericton, Canada.

Bandemer H.; Näther W. (1992): ***Fuzzy Data Analysis***. Kluwer Academic Publishers, Dordrecht.

Dubois D.; Prade H. (1980): ***Fuzzy Sets and Systems***. Academic Press, New York.

Hofmann-Wellenhof B.; Lichtenegger H.; Collins J. (1997): ***GPS Theory and Practice (4th Ed.).*** Springer, Wien New York.

Kutterer H. (2001a): ***Uncertainty assessment in geodetic data analysis***. In: Carosio A; Kutterer H. (Eds.): Proceedings of the First International Symposium on Robust Statistics and Fuzzy Techniques in Geodesy and GIS. Swiss Federal Institute of Technology (ETH) Zurich, Institute of Photogrammetry and Remote Sensing - Report No. 295, 7-12.

Kutterer H. (2001b): ***An imprecise search space for GPS phase ambiguity parameters***. In: Carosio A; Kutterer H. (Eds.): Proceedings of the First International Symposium on Robust Statistics and Fuzzy Techniques in Geodesy and GIS. Swiss Federal Institute of Technology (ETH) Zurich, Institute of Photogrammetry and Remote Sensing - Report No. 295, 215-220.

Kutterer (2002): ***Zum Umgang mit Ungewissheit in der Geodäsie - Bausteine für eine neue Fehlertheorie. Habilitation*** Thesis, DGK Series C 553, Munich.

Leinen S. (2001): ***Fuzzy-Logic based GPS On-The-Fly Ambiguity Search***. In: Carosio A; Kutterer H. (Eds.): Proceedings of the First International Symposium on Robust Statistics and Fuzzy Techniques in Geodesy and GIS. Swiss Federal Institute of Technology (ETH) Zurich, Institute of Photogrammetry and Remote Sensing - Report No. 295, 209-214.

Parkinson B. W.; Spilker J. J. (1996): ***Global Positioning System: Theory and Applications - Vols. I and II***. Progress in Astronautics and Aeronautics 163, American Institute of Astronautics and Aeronautics, Washington, DC.

Teunissen P. J. G.; Kleusberg A. (1998): ***GPS for Geodesy (2nd Ed.)***. Springer, Berlin Heidelberg.

Viertl R. (1996): ***Statistical Methods for Non-Precise Data. CRC Press***, Boca Raton New York London Tokyo.

Zadeh L. A. (1965): ***Fuzzy sets***. Information Control 8 (1965): 338 - 353.