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A Diagonalisation Algorithm and Its Application in Ambiguity Search

Guochang Xu

GeoForschungsZentrum Potsdam (GFZ), Dept. 1, Telegrafenberg, 14473 Potsdam, Germany

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Abstract. A diagonalisation algorithm of the least squares normal equation is proposed in this paper. The equivalent observation equations related to the diagonalised normal equations are also derived in detail. For the equivalent observation equations and their normal equations, the related equivalent ambiguity search criteria are outlined. Theoretical application of the proposed algorithm in ambiguity search is briefly summarised. Using this algorithm, the ambiguity search turns out to be a search in a diagonal space so that the search can be done very quickly. Numerical examples to illustrate the diagonalisation process of the normal equation and observation equation are also given.

Key words: GPS, Ambiguity Search, Diagonalisation Algorithm, Adjustment, Equivalent Equations

1 Introduction

It is well known that the ambiguity resolution is a key problem, which has to be solved in GPS static and kinematic precise positioning. Some well-derived ambiguity fixing and searching algorithms have been published during the last decades. A search process is usually needed in the most methods. Generally speaking, the search is an intensive computing and time consuming process. Therefore some optimal algorithms have been created to reduce the search area and to accelerate the search process (e.g. Euler and Landau, 1992; Han and Rizos, 1997; Teunissen, 1995). Using the well-known least squares ambiguity search criterion for searching in ambiguity domain, most modifications are directly based on the used criterion, e.g. via decomposition (fast method, e.g. Euler and Landau, 1992) or de-correlation (LAMBDA method, e.g. Teunissen, 1995). Alternatively, we will first try to work on equivalently diagonalised normal equations (and observation equations), and then use the related equivalent ambiguity search criteria for the equivalent problems. In this way the search process turns out to be a search in a diagonal space so that the time consuming on the search is negligible. This method is originally derived for the so-called general criterion (Xu, 2003) used in KSGSoft (Kinematic/Static GPS Software) (Xu et al., 1998); however, it may be directly used for the least squares ambiguity search criterion, too.

2 A Diagonalisation Algorithm of Least Squares Normal Equation

In least squares adjustment the unknowns can be divided into two groups and then solved in a block-wise manner. In practice, sometimes only one group of unknowns is of interest, the other group, called nuisance parameters, is better to be eliminated, for example because of the large size. The nuisance parameters can be eliminated block-wisely to obtain an equivalently eliminated normal equation system of the interested unknowns. Using the elimination process twice for the two groups of unknowns respectively, the normal equation can be diagonalised. The algorithm can be outlined as follows.

Linearized observation equation system can be represented by (Cui et al., 1982; Gotthardt, 1978; Zhou et al., 1997):

$$V = L - \begin{pmatrix} A_1 & A_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \qquad P \qquad (1)$$

where L is the observational vector of dimension m, A_1 and A_2 are coefficient matrices of dimension $m\times(n-r)$ and $m\times r$, X_1 and X_2 are unknown vectors of dimension n-r and r, V is residual vector of dimension m, n and m are numbers of total unknowns and observations respectively, P is a symmetric and definite weight matrix of dimension $m\times m$. For un-correlated observation vector L, P is a diagonal matrix.

Least squares normal equation of (1) can be formed then by

$$\begin{pmatrix} A_1 & A_2 \end{pmatrix}^T P \begin{pmatrix} A_1 & A_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \end{pmatrix}^T PL . \tag{2}$$

By denoting

$$\begin{pmatrix} A_1^T PA_1 & A_1^T PA_2 \\ A_2^T PA_1 & A_2^T PA_2 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = M$$

$$Inv(M) = Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \tag{3}$$

$$W_1 = A_1^T PL, \quad W_2 = A_2^T PL$$

the normal equation system (2) is written as

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}. \tag{4}$$

From the first equation of (4), one has

$$X_1 = Inv(M_{11})(W_1 - M_{12} X_2)$$

Set X_1 into the second equation of (4), one gets an equivalently eliminated normal equation of X_2

$$M_2 X_2 = B_2 , \tag{5}$$

where

$$\begin{aligned} M_2 &= M_{22} - M_{21} Inv(M_{11}) M_{12} \\ B_2 &= W_2 - M_{21} Inv(M_{11}) W_1 \end{aligned} \tag{6}$$

Similarly, from the second equation of (4), one has

$$X_2 = Inv(M_{22})(W_2 - M_{21} X_1) .$$

Set X_2 into the first equation of (4), one gets an equivalently eliminated normal equation of X_1

$$M_1 X_1 = B_1 , \tag{7}$$

where

$$\begin{aligned} M_1 &= M_{11} - M_{12} Inv(M_{22}) M_{21} \\ B_1 &= W_1 - M_{12} Inv(M_{22}) W_2 \end{aligned} \tag{8}$$

Combining (5) and (7), one has

$$\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \tag{9}$$

where (e.g. Cui et al., 1982; Gotthardt, 1978)

$$Q_{11} = Inv(M_1), \quad Q_{22} = Inv(M_2). \tag{10}$$

It is obvious that (4) and (9) are two equivalent normal equations. The solutions of the both equations are identical. Furthermore, Eq. (9) is a diagonalised normal equation related to the X_1 and X_2. So we call the process of computing (6) and (8) to form (5) and (7) (i.e. (9)) a diagonalisation process of the normal equation (4).

The diagonalisation process can be repeated $r-1$ times on the normal equation (5) (or the second equation of (9)), so that Equation (5) can be fully diagonalised and can be represented by

$$M_2' X_2 = B_2' \tag{11}$$

where M_2' is a diagonal matrix, r is the dimension of X_2, B_2' is a vector. Then (9) turns out to have a form of

$$\begin{pmatrix} M_1 & 0 \\ 0 & M_2' \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2' \end{pmatrix}. \tag{12}$$

Of course one may also diagonalise the X_1 related normal equation (7) if necessary. To be emphasized is that the normal equation system (12) is equivalent to the normal equation system (9) and (4). Through the diagonalisation process (12) is a diagonal equation system of X_1 and X_2. The sub-equation of X_2 is a diagonal sub-equation. A fully diagonalisation of the normal equation is indeed nearly the same as a Gauss-Jordan algorithm to eliminate the non-diagonal elements of the normal matrix of the normal equation and to solve the equation. The property of (12) could be very useful for many adjustment applications.

3 The Forming of Equivalent Observation Equations

Using the notations of (3), (6) can be rewritten as

$$\begin{aligned} M_2 &= A_2^T PA_2 - A_2^T PA_1 M_{11}^{-1} A_1^T PA_2 \\ &= A_2^T P(E - A_1 M_{11}^{-1} A_1^T P) A_2 \\ B_2 &= A_2^T PL - A_2^T PA_1 M_{11}^{-1} A_1^T PL \\ &= A_2^T P(E - A_1 M_{11}^{-1} A_1^T P) L \end{aligned} \tag{13}$$

where E is an identity matrix. Denoting (cf. Wang et al., 1988; Xu and Qian, 1986; Zhou, 1985)

$$J = A_1 M_{11}^{-1} A_1^T P \tag{14}$$

then one has the properties of

$$\begin{aligned} J^2 &= (A_1 M_{11}^{-1} A_1^T P)(A_1 M_{11}^{-1} A_1^T P) \\ &= A_1 M_{11}^{-1} A_1^T PA_1 M_{11}^{-1} A_1^T P \\ &= A_1 M_{11}^{-1} A_1^T P = J \end{aligned}$$

$$\begin{aligned} (E - J)(E - J) &= E^2 - 2EJ + J^2 \\ &= E - 2J + J = E - J \end{aligned}$$

$$\begin{aligned}[P(E-J)]^T &= (E-J^T)P \\ &= P-(A_1 M_{11}^{-1} A_1^T P)^T P \\ &= P - PA_1 M_{11}^{-1} A_1^T P \\ &= P(E-J)\end{aligned}$$

i.e., matrices J and $(E\text{-}J)$ are idempotent and $(E\text{-}J)^T P$ is symmetric, or

$$\begin{aligned}&J^2 = J, \qquad (E-J)^2 = E-J \\ &(E-J)^T P = P(E-J)\end{aligned} \tag{15}$$

Using above derived properties, (13) can be rewritten as:

$$\begin{aligned}M_2 &= A_2^T P(E-J)A_2 \\ &= A_2^T P(E-J)(E-J)A_2 \\ &= A_2^T (E-J)^T P(E-J)A_2 \\ B_2 &= A_2^T P(E-J)L = A_2^T (E-J)^T PL\end{aligned} \tag{16}$$

then the normal equation (5) can be rewritten as

$$\begin{aligned}&A_2^T (E-J)^T P(E-J)A_2 X_2 \\ &= A_2^T (E-J)^T PL\end{aligned} \tag{17}$$

Eq. (17) is the least squares normal equation of the following linear observation equation

$$U_2 = L-(E-J)A_2 X_2, \qquad P. \tag{18}$$

where L and P are original observation vector and weight matrix of (1), U_2 is a residual vector which has the same property as V in (1). Eq. (5) is the least squares normal equation of the equivalently eliminated observation equation (18).

Similarly, let

$$I = A_2 M_{22}^{-1} A_2^T P, \tag{19}$$

then Eq. (7) is the least squares normal equation of the following linear observation equation

$$U_1 = L-(E-I)A_1 X_1, \qquad P. \tag{20}$$

Again U_1 is a residual vector which has the same property as V in (1). Denote

$$D_1 = (E-I)A_1, \quad D_2 = (E-J)A_2, \tag{21}$$

Eq. (18) and (20) can be written together as

$$\begin{pmatrix}U_1 \\ U_2\end{pmatrix} = \begin{pmatrix}L \\ L\end{pmatrix} - \begin{pmatrix}D_1 & 0 \\ 0 & D_2\end{pmatrix}\begin{pmatrix}X_1 \\ X_2\end{pmatrix}, \quad \begin{pmatrix}P & 0 \\ 0 & P\end{pmatrix}. \tag{22}$$

Eq. (22) is derived from the normal equation (9), therefore, it is true inversely, i.e. (9) is the least squares normal equation of observation equation (22). Because the normal equations of (1) and (22) are (4) and (9), and (4) and (9) are equivalent, so (22) is an equivalent observation equation of (1). To be emphasised is that the equivalent observation equation (22) is a kind of diagonal equation of X_1 and X_2. This property is a direct derivation of the diagonal property of the normal equation (9).

Similarly, we may repeat above process $r-1$ times to the observation equation of X_2 (i.e. (18)) step by step and the related equivalent observation equation can be formed as

$$U_2' = L' - D_2' X_2, \quad P'. \tag{23}$$

where D_2' is a diagonal matrix, P' is a diagonal matrix of P, L' is a vector of L, U_2' is a residual vector which has the same property as V in (1). Then (22) turns out to have a form of

$$\begin{pmatrix}U_1 \\ U_2'\end{pmatrix} = \begin{pmatrix}L \\ L'\end{pmatrix} - \begin{pmatrix}D_1 & 0 \\ 0 & D_2'\end{pmatrix}\begin{pmatrix}X_1 \\ X_2\end{pmatrix}, \quad \begin{pmatrix}P & 0 \\ 0 & P'\end{pmatrix}. \tag{24}$$

(24) is equivalent to (22). Similarly, (12) is the normal equation of the observation equation (24).

Numerical examples are given in the appendix to illustrate the diagonalisation process of the normal equation and observation equation.

4 Equivalent Ambiguity Search Criteria

Suppose GPS observation equation is (1) and its least squares normal equation is (4), where $X_2 = N$ (N is the ambiguity sub-vector) and $X_1 = Y$ (Y is the rest unknown sub-vector). The least squares ambiguity search (LSAS) criterion (e.g., Teunissen, 1995; Leick, 1995; Hofmann-Wellenhof et al., 1997; Euler and Landau, 1992; Han and Rizos, 1997) is

$$\delta(dN) = (N_0 - N)Inv(Q_{22})(N_0 - N), \tag{25}$$

where N_0 is the float solution of the ambiguity sub-vector, $dN = N_0 - N$. The ambiguity search is a process to find out a vector N in the searching area so that the value of $\delta(dN)$ reaches the minimum. The general criterion is (Xu, 2003)

$$\delta(dX) = (X_0 - X)^T Inv(Q)(X_0 - X), \tag{26}$$

where $X = (Y \quad N)^T$, $X_0 = (Y_0 \quad N_0)^T$, $dX = X_0 - X$, index 0 denotes the float solution. The search in ambiguity domain is a process to find out a vector X (includes N in the searching area and Y computed) so that the value of $\delta(dX)$ reaches the minimum. The optimal property of this criterion is obvious (Xu, 2003).

For the equivalent observation equation (22), the related least squares normal equation is (9). The related least squares ambiguity search (LSAS) criterion remains the same as (25). The related general criterion is then (putting Q of (9) into (26) and taking (3) and (10) into account)

$$\begin{aligned}\delta_1(dX) &= (X_0 - X)^T\, Inv(Q)(X_0 - X)\\ &= (Y_0 - Y)^T\, Inv(Q_{11})(Y_0 - Y)\\ &\quad + (N_0 - N) Inv(Q_{22})(N_0 - N)\\ &= \delta(dY) + \delta(dN)\end{aligned} \tag{27}$$

where index 1 is used to distinguish the criterion (27) from the (26).

For the equivalent observation equation (24), the related least squares normal equation is (12). The related least squares ambiguity search (LSAS) criterion is then (putting Q_{22} of (12) into (25) and taking (3) into account)

$$\delta_2(dN) = (N_0 - N)^T M_2^{'} (N_0 - N)\,. \tag{28}$$

The related general criterion is then (putting Q of (12) into (26) and taking (3) and (10) into account)

$$\begin{aligned}\delta_2(dX) &= (X_0 - X)^T\, Inv(Q)(X_0 - X)\\ &= (Y_0 - Y)^T\, Inv(Q_{11})(Y_0 - Y)\\ &\quad + (N_0 - N)^T M_2^{'}(N_0 - N)\\ &= \delta(dY) + \delta_2(dN)\end{aligned} \tag{29}$$

where index 2 is used to distinguish the criteria (28) and (29) from the (25) and (27).

To be emphasised is that the observation equations (1), (22) and (24) are equivalent, and the related normal equations (4), (9) and (12) are also equivalent. Therefore, the LSAS criteria (25) and (28) are equivalent. The general criteria (26), (27) and (28) are equivalent too.

It is obvious that the ambiguity search should be based on the equivalent observation equations and the related normal equations due to their diagonal properties. Because of the diagonal property of the matrix $M_2^{'}$, the ambiguity search using criteria (29) or (28) turns out to be a search in a diagonal space and can be done very quickly.

5 Summary

The diagonalisation algorithm of the least squares normal equation proposed here may surely find interesting applications in the adjustments and data processing. The way of forming of the equivalent observation equation is also significant. Using the equivalent criteria based on the equivalent observation equations and the related equivalent normal equations, the ambiguity search can be made alternatively in a diagonal space. The diagonalisation process of the normal equation is nearly the same as the Gauss-Jordan algorithm to eliminate the non-diagonal elements of the normal matrix of the normal equation and to solve the normal equation. In other words, the diagonalisation process is almost the same as the process of solving the normal equation. Therefore the time consuming of the searching process is minimal and negligible. The numerical examples of the diagonalisation algorithm are given in the appendix.

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References

Brown R.G. and P.Y.C. Hwang (1992) ***Introduction of Random Signals and Applied Kalman Filter***, Second Edition, John Wiley & Sons, Inc., New York

Bauer M. (1994) ***Vermessung und Ortung mit Satelliten, Wichmann Verlag, Karslruhe***.

Cannon M.E., Lachapelle G., Szarmes M., Herbert J., Keith J. and Jokerst S. (1997) ***DGPS Kinematic Carrier Phase Signal Simulation Analysis for Precise Velocity and Position Determination***, Proceedings of ION NTM 97, Santa Monica, CA.

Cui X., Yu Z., Tao B. and Liu, D. (1982) ***Adjustment in Surveying,*** Surveying Publishing House, Peking.

Euler H.J., and Landau H. (1992) ***Fast GPS ambiguity resolution on-the-fly for real-time applications***, Proceedings of 6th Int. Geod. Symp. on satellite Positioning, Columbus, Ohio, 17-20.

Gotthardt E. (1978) ***Einfuehrung in die Ausgleichungsrechnung,*** Herbert Wichmann Verlag Karlsruhe.

Han S. and Rizos C. (1995) ***On-The-Fly Ambiguity Resolution for Long Range GPS Kinematic Positioning***, IAG Symposia 115, edited by Beutler, Hein, Melbourne and Seeber.

Han S. and Rizos, C. (1997) ***Comparing GPS Ambiguity Resolution Techniques***, GPS World, Oct. 1997, pp54-61.

Hofmann-Wellenhof B., Lichtenegger H. and Collins, J. (1997) ***GPS Theory and Practice***, Springer-Verlag, Wien

King R.W., Masters E.G. Rizos C., Stolz, A. and Collins J. (1987) ***Surveying with Global Positioning System*** **FERD**, Duemmler Verlag, Bonn.

Leick A. (1995) ***GPS Satellite Surveying***, New York: John Wiley & Sons.

Parkinson B.W., Spilker J.J. (Eds) (1996) ***Global Positioning System: Theory and Applications***, Volume I, II, Progress in Astronautics and Aeronautics, Volume 163

Remondi B. (1984) ***Using the Global Positioning System (GPS) phase observable for relative geodesy: modelling, processing, and results***, University of Texas at Austin, Center for Space Research.

Seeber G. (1993) ***Satellitengeodäsie***, Walter de Gruyter

Strang G. and Borre K. (1997) ***Linear Algebra, Geodesy, and GPS***, Wellesley-Cambridge Press

Teunissen P.J.G. (1995) ***The least-squares ambiguity decorrelation adjustment: a method for fast GPS integer ambiguity estimation***, Journal of Geodesy, 70, 65-82.

Wang G., Chen Z., Chen, W. and Xu G. (1988) ***The Principle of the GPS Precise Positioning System***, Surveying Publishing House, Peking

Xu G., Schwintzer P., Reigber Ch. (1998) ***KSGSoft (Kinematic/Static GPS Software) --- Software User Manual***, Scientific Technical Report 19/1998, GeoForschungsZentrum Potsdam

Xu G., Qian Z. (1986) ***The Application of Block Elimination Adjustment Method for Processing of the VLBI Data***, Crustal Deformation and Earthquake, Vol.6, No.4, in Chinese

Xu G. (2003) ***GPS – Theory, Algorithms and Applications***, Springer Verlag, Heidelberg

Zhou J. (1985) On the Jie factor, ***Acta Geodaetica et Geophysica***, No.5, in Chinese

Zhou J., Huang J., Yang Y. and Ou, J. (1997) ***Robust least squares method***, Publishing House of Huazhong University of Science and Technology, Wuhan

Appendix:

Numerical Examples of the Diagonalisation Algorithm

As discussed, a normal equation can be diagonalised and the related observation equation can be formed. Numerical examples to illustrate the diagonalisation process of the normal equation and observation equation are given below.

1). The Case of Two Variables

For observation equation (where σ is set to 1 which does not affect all results)

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$$P = \frac{1}{\sigma^2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{(a.1)}$$

the least squares normal equation is

$$\begin{pmatrix} 3 & 4 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}. \qquad \text{(a.2)}$$

Because

$$M_1 = 3 - 4(1/6)4 = 1/3, B_1 = 2 - 4(1/4)4 = -1/3,$$

$$M_2 = 6 - 4(1/3)4 = 2/3, B_2 = 4 - 4(1/3)2 = 4/3,$$

(a.2) is diagonalised as

$$\begin{pmatrix} 1/3 & 0 \\ 0 & 2/3 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 4/3 \end{pmatrix}. \qquad \text{(a.3)}$$

The solution $(X_1 = -2, X_2 = 2)$ of (a.3) is the same as that of (a.2). Furthermore, to form the equivalent observation equation, there are

$$M_{11} = A_1^T A_1 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 3,$$

$$M_{22} = A_2^T A_2 = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 6$$

$$I = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \frac{1}{6} \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

$$J = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$D_1 = (E - I) A_1 = \frac{1}{6} \begin{pmatrix} 5 & -2 & -1 \\ -2 & 2 & -2 \\ -1 & -2 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix},$$

$$D_2 = (E - J) A_2 = \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix},$$

thus (a.3) related observation equation is

$$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \left(\begin{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \end{pmatrix} - \begin{pmatrix} \frac{1}{3}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} & 0_{3\times1} \\ 0_{3\times1} & \frac{1}{3}\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \end{pmatrix} \right) \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$
$$\begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} \tag{a.4}$$

The normal equation of the observation (a.4) is exactly the (a.3). This numerical example shows that the normal equation and the related observation equation can be diagonalised.

2). The Case of Three Variables

For observation equation (where σ is set to 1 which does not affect all results)

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}, \tag{a.5}$$
$$P = \frac{1}{\sigma^2} E_{4\times4}$$

the least squares normal equation is

$$\begin{pmatrix} 7 & 5 & 6 \\ 5 & 4 & 5 \\ 6 & 5 & 7 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}. \tag{a.6}$$

Because of

$$M_{22}^{-1} = \begin{pmatrix} 4 & 5 \\ 5 & 7 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 7 & -5 \\ -5 & 4 \end{pmatrix}, \quad M_{11}^{-1} = \frac{1}{7},$$
$$M_1 = 7 - \begin{pmatrix} 5 & 6 \end{pmatrix} \frac{1}{3}\begin{pmatrix} 7 & -5 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \frac{2}{3},$$
$$B_1 = 2 - \begin{pmatrix} 5 & 6 \end{pmatrix} \frac{1}{3}\begin{pmatrix} 7 & -5 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{2}{3},$$
$$M_2 = \begin{pmatrix} 4 & 5 \\ 5 & 7 \end{pmatrix} - \begin{pmatrix} 5 \\ 6 \end{pmatrix}\frac{1}{7}\begin{pmatrix} 5 & 6 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 3 & 5 \\ 5 & 13 \end{pmatrix}$$
$$B_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 5 \\ 6 \end{pmatrix}\frac{1}{7}\cdot 2 = \frac{1}{7}\begin{pmatrix} 3 \\ 5 \end{pmatrix} \tag{a.7}$$

(a.6) is diagonalised as

$$\begin{pmatrix} 2/3 & 0 & 0 \\ 0 & 3/7 & 5/7 \\ 0 & 5/7 & 13/7 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -3/7 \\ -5/7 \end{pmatrix}. \tag{a.8}$$

The X_2 and X_3 related normal equation can be further diagonalised. Because of

$$M_1' = 3/7 - 5 \cdot (1/13) \cdot (5/7) = 2/13,$$
$$B_1' = -3/7 - 5 \cdot (1/13) \cdot (-5/7) = -2/13,$$
$$M_2' = 13/7 - 5 \cdot (1/3) \cdot (5/7) = 2/3,$$
$$B_2' = -5/7 - 5 \cdot (1/3) \cdot (-3/7) = 0,$$

(a.8) is further diagonalised as

$$\begin{pmatrix} 2/3 & 0 & 0 \\ 0 & 2/13 & 0 \\ 0 & 0 & 2/3 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -2/13 \\ 0 \end{pmatrix}. \tag{a.9}$$

The solution ($X_1 = 1$, $X_2 = -1$, $X_3 = 0$) of (a.9) is the same as that of (a.6) and (a.8). Furthermore, to form the equivalent observation equation of (a.8), there are

$$I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{pmatrix} \frac{1}{3}\begin{pmatrix} 7 & -5 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \end{pmatrix}$$
$$= \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix},$$

$$J = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} \frac{1}{7}\begin{pmatrix} 1 & 2 & 1 & 1 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \\ 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{pmatrix},$$

$$D_1 = (E - I)A_1 = \frac{1}{3}\begin{pmatrix} -1 \\ 2 \\ 0 \\ -1 \end{pmatrix},$$

$$D_2 = (E - J)A_2 = \frac{1}{7}\begin{pmatrix} 2 & 1 \\ -3 & -5 \\ 2 & 8 \\ 2 & 1 \end{pmatrix},$$

thus (a.8) related observation equation is

$$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} L \\ L \end{pmatrix} - \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \tag{a.10}$$

$$\begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}, \quad where \quad L = \begin{pmatrix} 2 \\ 1 \\ 0 \\ -2 \end{pmatrix}$$

The X_2 and X_3 related observation can be further diagonalised as follows. Because

$$I' = \frac{1}{7}\begin{pmatrix} 1 \\ -5 \\ 8 \\ 1 \end{pmatrix} \frac{7}{13} \cdot \frac{1}{7} \begin{pmatrix} 1 & -5 & 8 & 1 \end{pmatrix}$$

$$= \frac{1}{91}\begin{pmatrix} 1 & -5 & 8 & 1 \\ -5 & 25 & -40 & -5 \\ 8 & -40 & 64 & 8 \\ 1 & -5 & 8 & 1 \end{pmatrix},$$

$$J' = \frac{1}{7}\begin{pmatrix} 2 \\ -3 \\ 2 \\ 2 \end{pmatrix} \frac{7}{3} \cdot \frac{1}{7} \begin{pmatrix} 2 & -3 & 2 & 2 \end{pmatrix}$$

$$= \frac{1}{21}\begin{pmatrix} 4 & -6 & 4 & 4 \\ -6 & 9 & -6 & -6 \\ 4 & -6 & 4 & 4 \\ 4 & -6 & 4 & 4 \end{pmatrix},$$

$$D_2' = A_1' - I'A_1' = \frac{1}{7}\begin{pmatrix} 2 \\ -3 \\ 2 \\ 2 \end{pmatrix} - \frac{1}{7 \cdot 91}\begin{pmatrix} 35 \\ -175 \\ 280 \\ 35 \end{pmatrix}$$

$$= \frac{1}{13}\begin{pmatrix} 3 \\ -2 \\ -2 \\ 3 \end{pmatrix}$$

$$D_3' = A_2' - J'A_2' = \frac{1}{7}\begin{pmatrix} 1 \\ -5 \\ 8 \\ 1 \end{pmatrix} - \frac{1}{21 \cdot 7}\begin{pmatrix} 70 \\ -105 \\ 70 \\ 70 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \end{pmatrix}$$

thus (a.9) related observation equation is

$$\begin{pmatrix} U_1 \\ U_2' \\ U_3' \end{pmatrix} = \begin{pmatrix} L \\ L \\ L \end{pmatrix} - \begin{pmatrix} D_1 & 0 & 0 \\ 0 & D_2' & 0 \\ 0 & 0 & D_3' \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \tag{a.11}$$

$$\begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{pmatrix}$$

The normal equation (a.6) and its related observation equation (a.5) are fully diagonalised as (a.9) and (a.11), respectively. These numerical examples show that the normal equation and the related observation equation can be diagonalised as described.